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Von Neumann's Dynamic General Equilibrium
Restated and Solved by Elementary Algebra

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January 1984

Von Neumann's Dynamic General Equilibrium Restated and Solved by Elementary Algebra

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VON NEUMANN'S DYNAMIC GENERAL EQUILIBRIUM

RESTATED AND SOLVED BY ELEMENTARY ALGEBRA

Hans Brems

Abstract

Walras believed his general economic equilibrium to be determinate in the sense that the number of equations entailed is equal to the number of unknowns.

Such innocence lasted until the 1930's. In a new breakthrough John von Neumann first formulated a balanced and steady-state growth of a general economic equilibrium and proved the existence of a solution.

The breakthrough did not lie in the subject matter, which was still allocation and relative price in general equilibrium using maxima and minima. What was new was method rather than subject matter. The maxima and minima were handled without the use of any calculus at all. What von Neumann taught us was to use inequalities to formulate a primal and a dual problem. What von Neumann offered was a solution of his primal and dual problem displaying a saddle point: the maximized rate of growth equalled the minimized rate of interest.

One of the foremost mathematicians of the twentieth century, von Neumann used nonelementary algebra in his proof of the existence of a competitive equilibrium. The purpose of the present paper is to show that if collapsed into two goods and two processes, the von Neumann model may be solved by elementary algebra.

Since the von Neumann system is homogeneous of degree zero in its process levels, excess supplies, goods prices, and loss margins, solutions were possible only for <u>relative</u> process levels, excess supplies, goods prices, and loss margins.

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VON NEUMANN'S DYNAMIC GENERAL EQUILIBRIUM
RESTATED AND SOLVED BY ELEMENTARY ALGEBRA

Hans Brems

> The paper contains the first explicit statement, known to this author, of what has subsequently been called the activity analysis model of production. ... Finally, ... the paper contains the first rigorous, formal, and fully explicit model in nonaggregative capital theory known to this author.
>
> Tjalling C. Koopmans (1964: 356)

1. Von Neumann's Problem

Walras [1874-1877 (1954: 43-44)] believed his general equilibrium to be determinate "in the sense that the number of equations entailed is equal to the number of unknowns." For the next sixty years, as pointed out by the younger Menger (1971: 50), Walras's belief remained unquestioned. Neither uniqueness nor feasibility was ever discussed.

On the European Continent general equilibrium was best known in the form of Cassel's [1923 (1932: 32-41 and 152-155)] dynamized formulation of it, "the uniformly progressing state." Like Walras, Cassel had allowed for substitution in consumption but had failed to allow for it

in production, had failed to treat the distinction between free and economic goods as endogeneous, and had failed to prove the existence of a solution.

Such innocence lasted until the 1930's. In a new breakthrough John von Neumann [1937 (1945-1946)] first formulated a balanced and steady-state growth of a general economic equilibrium and proved the existence of a solution. The breakthrough did not lie in the subject matter, which was still allocation and relative price in general equilibrium using maxima and minima. Indeed all economists can appreciate the simple beauty, yet high degree of generality, characterizing the von Neumann model. There is substitution both in production and consumption. The model "can handle capital goods without fuss and bother," as Dorfman-Samuelson-Solow (1958) put it. There is explicit optimization in the model: the solution weeds out all but the most profitable process or processes. There are free and economic goods, indeed the solution tells us which will be free and which economic.

2. Von Neumann's Method

What was new was method rather than subject matter. This time, from the very beginning, the matter was in the hands of mathematicians and remained there, and the mathematics deployed was very different

from the calculus deployed after 1870. The maxima and minima were handled without the use of any calculus at all. What von Neumann taught us was to use inequalities to formulate a primal and a dual problem. What von Neumann offered was a solution of his primal and dual problem displaying a saddle point.

3. Our Restatement of von Neumann

We must convey the flavor of von Neumann's method. But being one of the foremost mathematicians of the twentieth century, von Neumann used nonelementary algebra. So do Kemeny, Snell, and Thompson (1957) and Lancaster (1968) in their excellent restatements of von Neumann. Can the von Neumann model be solved by elementary algebra? If collapsed into two goods and two processes, it can, and we shall show how.

I. NOTATION

1. Variables

$g \equiv$ proportionate rate of growth

$P_i \equiv$ price of ith good

$r \equiv$ rate of interest

$u_i \equiv$ excess supply of ith good

$v_j \equiv$ loss margin of jth process

$X_j \equiv$ level of jth process

2. Parameters

$a_{ij} \equiv$ input of ith good absorbed per unit of jth process level

$b_{ij} \equiv$ output of ith good supplied per unit of jth process level

II. THE MODEL

1. Processes, Their Level, and Their Rate of Growth

A von Neumann good may be absorbed as an input as well as supplied as an output. A von Neumann process may have several inputs and several outputs, and its unit level may arbitrarily be defined as the unit of any one input or any one output per unit of time.

Let there be m goods and n processes. Operated at unit level, the jth process converts a_{1j}, ..., a_{mj} units of the m goods absorbed as inputs into b_{1j}, ..., b_{mj} units of the m goods supplied as output. The coefficients a_{ij} and b_{ij} are nonnegative technological parameters, but let each process have at least one positive a_{ij}, i.e., be absorbing at least one good as an input. And let each good have at least one positive b_{ij}, i.e., be supplied as an output in at least one process. The level of the jth process is the pure number X_j by which unit level should be multiplied in order to get actual input or output. The proportionate rate of growth g_j of the level of the jth process is defined

$$X_j(t + 1) \equiv [1 + g_j(t)]X_j(t) \tag{1}$$

A von Neumann process can handle joint supply of and demand for goods, indeed consists of such supply and demand. Yet the von Neumann model can handle substitution in both production and consumption. First, there is substitution in production, for although each process has parametric input coefficients a_{ij} and output coefficients b_{ij}, the same good may occur as an output in more than one process, hence may be produced in more than one way. Second, there is substitution in consumption, for labor is a good like any other, hence is reproducible:

Labor is simply the output of one or more processes whose inputs are consumers' goods. Although each such process has parametric input coefficients a_{ij} and output coefficients b_{ij}, labor may occur as an output in more than one process, hence may be produced in more than one way--by being fed, so to speak, alternative menus.

Does the von Neumann model have capital in it? It does, in fact it incorporates the time element of production in a particularly elegant way. In the von Neumann model all processes have a period of production of one time unit, but this is less restrictive than it sounds: As for circulating capital, if consumable wine has a period of production of two years, simply define two distinct processes and goods as follows. The first process absorbs zero-year old wine and supplies one-year old wine; the second absorbs one-year old wine and supplies two-year old wine. As for fixed capital, if the useful life of machines is two years, again define two distinct processes and goods. The first process absorbs zero-year old machines and supplies one-year old machines; the second absorbs one-year old machines and supplies two-year old machines!

Since a process absorbing its inputs at time t supplies its output at time t + 1, should the time coordinate of its level be that of its input or that of its output? Arbitrarily let it be the latter.

2. Excess Demand Must Be Nonpositive

Let the level of the jth process be $X_j(t + 1)$. As a result, the input of the ith good absorbed at time t is $a_{ij}X_j(t + 1)$. Let the level of the jth process be $X_j(t)$. As a result, the output of the ith good supplied at time t is $b_{ij}X_j(t)$. We may then use (1) and write excess demand for the ith good in the jth process at time t as

$$a_{ij}X_j(t + 1) - b_{ij}X_j(t) = \{a_{ij}[1 + g_j(t)] - b_{ij}\}X_j(t)$$

which will be positive, zero, or negative as the brace of the right-hand side is positive, zero, or negative. Now some processes may have positive, some zero, and some negative excess demand for the ith good. But feasibility requires overall excess demand to be nonpositive. The sum of all inputs of the ith good absorbed in all processes must be smaller than or equal to the sum of all outputs of it supplied in all processes:

$$a_{i1}X_1(t + 1) + \ldots + a_{in}X_n(t + 1) \leq b_{i1}X_1(t) + \ldots + b_{in}X_n(t) \qquad (2)$$

where i = 1, ..., m. If for the ith good the less-than sign applies, that good at time t is a free good having a zero price: $P_i(t) = 0$. Rule out the uninteresting case that all goods are free and assume

that at least one is not, i.e., that in the system (2) at least one equality sign applies.

3. Profits Must be Nonpositive

At time t + 1 let the jth process be operated at unit level. The inputs absorbed at time t at unit level are a_{ij}, where $i = 1, \ldots, m$. Such inputs are purchased at the prices $P_i(t)$. Hence the input costs at unit level are $a_{ij}P_i(t)$ and their sum is $a_{1j}P_1(t) + \ldots + a_{mj}P_m(t)$. The outputs supplied at time t + 1 at unit level are b_{ij}. Such outputs are sold at prices $P_i(t + 1)$. Hence the revenues at unit level are $b_{ij}P_i(t + 1)$, and their sum is $b_{1j}P_1(t + 1) + \ldots + b_{mj}P_m(t + 1)$.

Now under pure competition and freedom of entry and exit, profits must be nonpositive, hence for the jth process the sum of all input cost at time t with interest added at the rate r must be greater than or equal to the sum of all revenue at time t + 1:

$$[1 + r(t)][a_{1j}P_1(t) + \ldots + a_{mj}P_m(t)] \geqq b_{1j}P_1(t + 1) + \ldots + b_{mj}P_m(t + 1) \quad (3)$$

where $j = 1, \ldots, n$. If for the jth process the greater-than sign applies, that process at time t + 1 is a money-losing one to be operated at zero level: $X_j(t + 1) = 0$. Rule out the uninteresting

case that all processes are money-losing ones and assume that at least one is not, i.e., that in the system (3) at least one equality sign applies.

4. Equilibrium

Cassel had thought of equilibrium growth as balanced steady-state growth of all physical outputs. Von Neumann thought of equilibrium growth as balanced steady-state growth of all process levels. Balanced growth of process levels means that the proportionate rates of growth of all process levels are equal:

$$g_1(t) = \ldots = g_n(t) \tag{4}$$

Steady-state growth of process levels means that the proportionate rate of growth of all process levels are stationary:

$$g_j(t + 1) = g_j(t); \qquad j = 1, \ldots, n \tag{5}$$

Like Cassel, von Neumann also required the rate of interest and all prices to be stationary:

$$r(t + 1) = r(t) \tag{6}$$

$$P_i(t + 1) = P_i(t); \qquad i = 1, \ldots, m \tag{7}$$

5. The Primal Problem: Maximize the Rate of Growth

In inequality (2) use (1) to express all $X_j(t + 1)$ in terms of $X_j(t)$. Use (4) to strip $g_j(t)$ of all its subscripts and (5) to strip it of its time coordinate and write (2) as

$$(1 + g)[a_{i1}X_1(t) + \ldots + a_{in}X_n(t)] \leqq b_{i1}X_1(t) + \ldots + b_{in}X_n(t) \tag{8}$$

where $i = 1, \ldots, m$. The system (8) expresses the growth pattern of goods: If the less-than sign of (8) applies, the economy more than reproduces what it absorbed one period earlier of the ith good raised by the growth rate g, hence the ith good is growing at a rate higher than g. If the equality sign of (8) applies, the ith good is growing at the rate g.

We can always make the rate of growth g high enough to generate positive excess demand for at least one good. But how high can we make it <u>without</u> doing that? When the rate of growth reaches its highest possible value, i.e., its equilibrium value, excess demand

will become zero for at least one good. That good or those goods will then become economic. All other goods will be growing more rapidly and become free. Consequently we may indeed assume, as we did, that at least one equality sign applies. The equilibrium rate of growth g will then be the rate of growth of the slowest-growing good or goods. Goods growing more rapidly than that become free.

Thus von Neumann had formulated his primal problem: maximize the rate of growth g subject to the constraint (8).

Notice that while process-level growth is balanced, goods growth is unbalanced!

6. The Dual Problem: Minimize the Rate of Interest

In Inequality (3) use (6) and (7) to strip $r(t)$ and $P_i(t)$ of their time coordinates and write it as

$$(1 + r)(a_{1j}P_1 + \ldots + a_{mj}P_m) \geqq b_{1j}P_1 + \ldots + b_{mj}P_m \tag{9}$$

where $j = 1, \ldots, n$. The system (9) expresses the profitability pattern of processes: If the greater-than sign of (9) applies, revenue from the process falls short of its cost one period earlier with

interest added to it at the rate r, hence the process is losing money. If the equality sign of (9) applies, the process is breaking even.

We can always make the rate of interest r low enough to generate positive profits in at least one process. But how low can we make it without doing that? When the rate of interest reaches its lowest possible value, i.e., its equilibrium value, profits will become zero in at least one process. That process or those processes will then break even and be operated. All other processes will be money-losing and remain unused. Consequently we may indeed assume, as we did, that at least one equality sign applies. The equilibrium rate of interest will then be the internal rate return of the most profitable process or processes. Processes less profitable than that will remain unused.

Thus von Neumann had formulated his dual problem: minimize the rate of interest subject to the constraint (9).

III. SOLUTION

1. Collapsing the Model to Two Goods and Two Processes

Using nonelementary algebra, von Neumann proved the existence of an equilibrium solution displaying a saddle point: the maximized rate of growth equalled the minimized rate of interest. But if we collapse the von Neumann model to two goods and two processes, elementary algebra will do to establish such a solution--as we shall now show.

2. Nonpositive Excess Demand Expressed As an Equality

Since in (8) all variables refer to the same time let us suppress its time coordinates. By introducing a nonnegative auxiliary variable $u_i \geqq 0$ we may write (8) as an equality rather than as an inequality:

$$(1 + g)(a_{i1}X_1 + a_{i2}X_2) + u_i = b_{i1}X_1 + b_{i2}X_2$$

or

$$u_i = b_{i1}X_1 + b_{i2}X_2 - (1 + g)(a_{i1}X_1 + a_{i2}X_2) \tag{10}$$

from which the economic meaning of u_i is seen to be current physical output <u>minus</u> current physical input of ith good or simply excess supply of ith good. Feasibility required excess demand of ith good to be nonpositive, hence requires the excess supply of it u_i to be nonnegative.

3. <u>Nonpositive Profits Expressed As an Equality</u>

By introducing a nonnegative auxiliary variable $v_j \geq 0$ we may write (9) as an equality rather than as an inequality:

$$(1 + r)(a_{1j}P_1 + a_{2j}P_2 = b_{1j}P_1 + b_{2j}P_2 + v_j$$

or

$$v_j = (1 + r)(a_{1j}P_1 + a_{2j}P_2) - (b_{1j}P_1 + b_{2j}P_2) \tag{11}$$

from which the economic meaning of v_j is seen to be unit-level cost with interest <u>minus</u> unit-level revenue or simply loss margin of jth process. Freedom of entry and exit required the profit margin of the

jth process to be nonpositive, hence requires its loss margin v_j to be nonnegative.

4. The Saddle Point: Maximized Rate of Growth Equals Minimized Rate of Interest

Multiply the excess supply (10) of the ith good by its price P_i, and write out the result for both goods, i = 1, 2. Multiply the loss margin (11) of the jth process by its level X_j, and write out the result for both processes j = 1, 2. The four equations are

$$P_1u_1 = [b_{11} - (1 + g)a_{11}]P_1X_1 + [b_{12} - (1 + g)a_{12}]P_1X_2 \tag{12}$$

$$P_2u_2 = [b_{21} - (1 + g)a_{21}]P_2X_1 + [b_{22} - (1 + g)a_{22}]P_2X_2 \tag{13}$$

$$v_1X_1 = [(1 + r)a_{11} - b_{11}]P_1X_1 + [(1 + r)a_{21} - b_{21}]P_2X_1 \tag{14}$$

$$v_2X_2 = [(1 + r)a_{12} - b_{12}]P_1X_2 + [(1 + r)a_{22} - b_{22}]P_2X_2 \tag{15}$$

Then add the four equations (12) through (15) and find

$$P_1u_1 + P_2u_2 + v_1X_1 + v_2X_2 =$$

$$(r - g)(a_{11}P_1X_1 + a_{12}P_1X_2 + a_{21}P_2X_1 + a_{22}P_2X_2) \tag{16}$$

But if excess supply u_i of the ith good is zero, price P_i is positive, and if excess supply u_i is positive, price P_i is zero. Consequently the product P_iu_i always has one and only one factor equalling zero and must itself be zero. Likewise, if loss margin v_j of the jth process is zero, process level X_j is positive, and if loss margin v_j is positive, process level X_j is zero. Consequently the product v_jX_j, too, always has one and only one factor equalling zero and must itself be zero, and the entire left-hand side of (16) is zero.

As a result, at least one of the factors on the right-hand side must be zero. Now we have ruled out the uninteresting case that all goods are free and assumed that at least one is not, i.e., that in the system (2) at least one equality sign applies meaning that at least one excess supply u_i is zero. Since the numbering of goods is arbitrary, we may let that u_i be $u_2 = 0$; consequently the second good has a positive price: $P_2 > 0$. We have also ruled out the uninteresting case that all processes are money-losing ones and assumed that at least one is not, i.e., that in the system (3) at least one equality sign applies meaning that at least one loss margin v_j is

zero. Since the numbering of processes is also arbitrary, we may let that v_j be $v_2 = 0$; consequently the second process has a positive level: $X_2 > 0$. But if both $P_2 > 0$ and $X_2 > 0$, then in the last factor of (16) at least the product $P_2X_2 > 0$. Furthermore, for the second good to be economic, some of it must be absorbed as an input in the second process, possibly the only one operated, i.e., $a_{22} > 0$. Consequently $a_{22}P_2X_2 > 0$, and the only way the right-hand side of (16) can be zero is if

$$g = r \tag{17}$$

Using elementary algebra we have proved the existence of an equilibrium solution displaying a saddle point: the maximized rate of growth equals the minimized rate of interest.[1]

5. Solutions for Process Levels and Goods Prices

We may now find solutions for process levels and goods prices. As a first step, in (10) set $u_2 = 0$ and write the equation for both goods, $i = 1, 2$:

$$1 + g = \frac{b_{11}X_1 + b_{12}X_2 - u_1}{a_{11}X_1 + a_{12}X_2} \tag{18}$$

$$1 + g = \frac{b_{21}X_1 + b_{22}X_2}{a_{21}X_1 + a_{22}X_2} \tag{19}$$

As a second step, in (11) set $v_2 = 0$ and write the equation for both processes, $j = 1, 2$:

$$1 + r = \frac{b_{11}P_1 + b_{21}P_2 + v_1}{a_{11}P_1 + a_{21}P_2} \tag{20}$$

$$1 + r = \frac{b_{12}P_1 + b_{22}P_2}{a_{12}P_1 + a_{22}P_2} \tag{21}$$

Eqs. (18) through (21) would remain the same if process levels X_j, excess supplies u_i, goods prices, P_i, and loss margins v_j were all multiplied by an arbitrary positive constant λ. Consequently the von Neumann

system is homogeneous of degree zero in those four variables--unlike the Walras system which was homogeneous of degree zero only in its prices, money expenditures, and money incomes. Our only hope, then, is to solve for relative process levels, excess supplies, goods prices, and loss margins. Like Walras, we must choose numéraires. We begin by choosing numéraires that will guarantee the meaningfulness of such relative variables.

6. Relative Process Levels Are Meaningful

Since the numbering of processes is arbitrary, we assumed $X_2 > 0$. In that case division by X_2 is meaningful. Divide numerators and denominators alike of (18) and (19) by X_2, define relative process level $x \equiv X_1/X_2$, and write (18) and (19) in terms of x and u_1/X_2:

$$1 + g = \frac{b_{11}x + b_{12} - u_1/X_2}{a_{11}x + a_{12}} \tag{22}$$

$$1 + g = \frac{b_{21}x + b_{22}}{a_{21}x + a_{22}} \tag{23}$$

Set the right-hand sides of (22) and (23) equal, multiply across, and arrive at the quadratic in x:

$$x^2 + \frac{a_{11}b_{22} + a_{12}b_{21} - a_{21}b_{12} - a_{22}b_{11} + a_{21}u_1/X_2}{a_{11}b_{21} - a_{21}b_{11}} x$$

$$+ \frac{a_{12}b_{22} - a_{22}b_{12} + a_{22}u_1/X_2}{a_{11}b_{21} - a_{21}b_{11}} = 0 \qquad (24)$$

7. Relative Prices Are Meaningful

Since the numbering of goods is arbitrary, we assumed $P_2 > 0$. In that case division by P_2 is meaningful. Divide numerators and denominators alike of (20) and (21) by P_2, define relative price $p \equiv P_1/P_2$, and write (20) and (21) in terms of p and v_1/P_2:

$$1 + r = \frac{b_{11}p + b_{21} + v_1/P_2}{a_{11}p + a_{21}} \qquad (25)$$

$$1 + r = \frac{b_{12}p + b_{22}}{a_{12}p + a_{22}} \qquad (26)$$

Set the right-hand sides of (25) and (26) equal, multiply across, and arrive at the quadratic in p:

$$p^2 + \frac{a_{11}b_{22} - a_{12}b_{21} + a_{21}b_{12} - a_{22}b_{11} - a_{12}v_1/P_2}{a_{11}b_{12} - a_{12}b_{11}}\, p$$

$$+ \frac{a_{21}b_{22} - a_{22}b_{21} - a_{22}v_1/P_2}{a_{11}b_{12} - a_{12}b_{11}} = 0 \qquad (27)$$

8. Four Possibilities

As we saw, the products $P_i u_i$ and $v_j X_j$ always have one and only one factor equalling zero and must themselves be zero. As we assumed, $P_2 > 0$ and $X_2 > 0$, hence division by them is meaningful. Consequently

$$pu_1 = v_1 x = 0 \qquad (28)$$

Our results (24), (27), and (28) represent a quadratic system in four variables, i.e., relative process level x, relative excess supply u_1/X_2, relative price p, and relative loss margin v_1/P_2. Feasibility requires all four of them to be nonnegative. Depending upon our technology matrix a_{ij}, b_{ij} the roots of our quadratic system may or may not be feasible and when feasible may or may not be unique. Generally we find four possibilities: (1) One free good, one unused process; (2) one free good, no unused process; (3) no free good, one unused process; and (4) no free good, no unused process. Let us examine the technology matrix which will permit each of these possibilities to materialize.

9. First Possibility. One Free Good, One Unused Process: The Economics

For the second good to become the only economic good, two conditions must hold.

First, $b_{21}/a_{21} < b_{11}/a_{11}$ meaning that in the first process, if operated, the second good is growing less rapidly than the first good.

Second, $b_{22}/a_{22} < b_{12}/a_{12}$ meaning that in the second process, too, the second good is growing less rapidly than the first good.

Consequently, whether or not the first process is operated the second good is growing less rapidly than the first good, and there is

no way in which the two goods could be growing at the same rate. As a result, the first good will become free.

For the second process to become the only process operated, two conditions must hold.

Third, $b_{11}/a_{11} < b_{12}/a_{12}$ meaning that the first good, if economic, has a lower revenue-cost ratio in the first process than in the second process.

Fourth, $b_{21}/a_{21} < b_{22}/a_{22}$ meaning that the second good, too, has a lower revenue-cost ratio in the first process than in the second process.

Consequently, whether or not the first good is economic the first process is less profitable than the second, and there is no way in which the two processes could be earning the same internal rate of return. As a result, the first process will remain unused.

If our first and second conditions are met, the first good will become free anyway, and the third condition can be dispensed with. And if our third and fourth conditions are met, the first process will remain unused anyway, and the first condition can be dispensed with.

10. First Possibility. One Free Good, One Unused Process: The Algebra

Either our first, second, and fourth or our second, third, and fourth conditions are met if:

$$\frac{b_{21}}{a_{21}} < \frac{b_{11}}{a_{11}} < \frac{b_{22}}{a_{22}} < \frac{b_{12}}{a_{12}} \text{ or } \frac{b_{21}}{a_{21}} < \frac{b_{22}}{a_{22}} < \frac{b_{11}}{a_{11}} < \frac{b_{12}}{a_{12}} \text{ or}$$

$$\frac{b_{21}}{a_{21}} < \frac{b_{22}}{a_{22}} < \frac{b_{12}}{a_{12}} < \frac{b_{11}}{a_{11}} \text{ or } \frac{b_{11}}{a_{11}} < \frac{b_{21}}{a_{21}} < \frac{b_{22}}{a_{22}} < \frac{b_{12}}{a_{12}} \tag{29}$$

making it possible to have at the same time $p = 0$ and $x = 0$. But a zero price of the first good shows that the excess supply of it is positive: $u_1 > 0$, so the first part of the solution for the present case is the positive relative excess supply

$$u_1/X_2 = -\frac{a_{12}b_{22} - a_{22}b_{12}}{a_{22}} \tag{30}$$

which will make the third term of (24) zero, hence produce a root $x = 0$. What will the other root be like? Inserting (30) into (24) we find that in the second and third case of (29) the two roots of (24) are $x = 0$ and $x < 0$, so there is only one feasible root, i.e., the former. But in the first and fourth case of (29) the root other than $x = 0$ may be positive or negative, i.e., feasible or nonfeasible, respectively, and the uniqueness of a feasible root may be lost.

A zero level of the first process shows that its loss margin is positive: $v_1 > 0$, so the second part of the solution is the positive relative loss margin

$$v_1/P_2 = \frac{a_{21}b_{22} - a_{22}b_{21}}{a_{22}} \tag{31}$$

which will make the third term of (27) zero, hence produce a root $p = 0$. What will the other root be like? Inserting (31) into (27) we find that in the first and fourth case of (29) the two roots of (27) are $p = 0$ and $p < 0$, so there is only one feasible root, i.e., the former. But in the second and third case of (29) the root other than $p = 0$ may be positive or negative, i.e., feasible or nonfeasible, respectively, and the uniqueness of a feasible root may be lost.

In this case, what will the rates of growth and interest be? Well, as long as (30) holds, there is a root $x = 0$. So we may insert (30) into (22), set $x = 0$, and find

$$1 + g = b_{22}/a_{22} < b_{12}/a_{12} \tag{32}$$

or in English, the second (economic) good is growing less rapidly than the first (free) good in the only process being operated, i.e., the second one.

As long as (31) holds, there is a root $p = 0$. So we may insert (31) into (25), set $p = 0$, and find

$$1 + r = b_{22}/a_{22} > b_{21}/a_{21} \tag{33}$$

or, in English, had it produced the only economic good, i.e., the second one, the first process would have earned an internal rate of return less than that earned by producing that good in the only process being operated, i.e., the second one.

We defined the coefficients a_{ij} and b_{ij} as nonnegative technological parameters but assumed each process to have at least one positive a_{ij} and each good to have at least one positive b_{ij}. For the second good to be economic in the first place, some of it must be

absorbed as an input, i.e., $a_{22} > 0$. For the second process to be earning a return on it in the first place, the second process must be supplying it as an output, i.e., $b_{22} > 0$. If so, our solutions (32) and (33) are meaningful and positive.

11. Second Possibility. One Free Good, No Unused Process: The Economics

Could there be one free good but no unused process?

For the second good to become the only economic good, the same two conditions must hold as in our first possibility:

First, $b_{21}/a_{21} < b_{11}/a_{11}$ meaning that in the first process, if operated, the second good is growing less rapidly than the first good.

Second, $b_{22}/a_{22} < b_{12}/a_{12}$ meaning that in the second process, too, the second good is growing less rapidly than the first good.

Consequently, whether or not the first process is operated the second good is growing less rapidly than the first good, and there is no way in which the two goods could be growing at the same rate. As a result, the first good will become free.

But with only one economic good, both processes can be operated only if they are equally profitable in producing that good. Consequently $b_{21}/a_{21} = b_{22}/a_{22}$ meaning that the second good has the same revenue-cost ratio in the first process as in the second process.

12. Second Possibility. One Free Good, No Unused Process: The Algebra

All three conditions are met if

$$\frac{b_{21}}{a_{21}} = \frac{b_{22}}{a_{22}} < \frac{b_{11}}{a_{11}} < \frac{b_{12}}{a_{12}} \text{ or } \frac{b_{21}}{a_{21}} = \frac{b_{22}}{a_{22}} < \frac{b_{12}}{a_{12}} < \frac{b_{11}}{a_{11}} \tag{34}$$

The third term of (27) will still be zero, hence $p = 0$, if (31) holds. But according to (34) we now have $a_{21}b_{22} = a_{22}b_{21}$ which in (31) would make $v_1/P_2 = 0$, which is feasible. But a zero loss margin v_1 of the first process would mean that the level of that process would be positive: $x > 0$. In short, if in this case relative price p is zero, then relative process level x cannot be.

To find the internal rate of return common to the two processes operated write (11) for $j = 1$, divide it by P_2, assumed to be positive, and write relative loss margin of the first process

$$v_1/P_2 = [(1 + r)a_{11} - b_{11}]p + (1 + r)a_{21} - b_{21} \tag{35}$$

But in the present case $v_1/P_2 = 0$, and there is a root $p = 0$. As a result (35) collapses into

$$1 + r = b_{21}/a_{21} = b_{22}/a_{22} \tag{36}$$

or in English, both processes yield the same internal rate of return r for the only good priced positively, i.e., the second one.

13. Third Possibility. No Free Good, One Unused Process: The Economics

Could there be no free good but one unused process?

For the second process to become the only process operated, the same two conditions must hold as in our first possibility:

Third, $b_{11}/a_{11} < b_{12}/a_{12}$ meaning that the first good, if economic, has a lower revenue-cost ratio in the first process than in the second process.

Fourth, $b_{21}/a_{21} < b_{22}/a_{22}$ meaning that the second good, too, has a lower revenue-cost ratio in the first process than in the second process.

Consequently, whether or not the first good is economic the first process is less profitable than the second, and there is no way in which the two processes could be earning the same internal rate of return. As a result, the first process will remain unused.

But with only one process being operated, both goods can be economic only if they are growing at the same rate in that process. Consequently $b_{22}/a_{22} = b_{12}/a_{12}$ meaning that in the second process the second good is growing at the same rate as the first good.

14. Third Possibility. No Free Good, One Unused Process: The Algebra

All three conditions are met if

$$\frac{b_{11}}{a_{11}} < \frac{b_{21}}{a_{21}} < \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}} \text{ or } \frac{b_{21}}{a_{21}} < \frac{b_{11}}{a_{11}} < \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}} \qquad (37)$$

The third term of (24) will still be zero, hence $x = 0$, if (30) holds. But according to (37) we now have $a_{12}b_{22} = a_{22}b_{12}$ which in (30) would make $u_1/X_2 = 0$, which is feasible. But a zero excess supply u_1 of the first good would mean that the price of that good would be positive: $p > 0$. In short, if in this case relative process level x is zero, then relative price p cannot be.

To find the rate of growth common to the two economic goods write (10) for $i = 1$, divide it by X_2, assumed to be positive, and write relative excess supply of the first good

$$u_1/X_2 = [b_{11} - (1 + g)a_{11}]x + b_{12} - (1 + g)a_{12} \tag{38}$$

But in the present case $u_1/X_2 = 0$, and there is a root $x = 0$. As a result (38) collapses into

$$1 + g = b_{12}/a_{12} = b_{22}/a_{22} \tag{39}$$

or in English, both goods are growing at the same rate g in the only process being operated, i.e., the second one.

15. Fourth Possibility. No Free Good and No Unused Process: The Economics

Would it be possible for both goods to be economic?

Suppose $b_{22}/a_{22} < b_{21}/a_{21}$ meaning that the second good has a lower revenue-cost ratio in the second process than in the first process. So if only the second good were economic the second process would have a lower revenue-cost ratio than the first process.

But also suppose $b_{11}/a_{11} < b_{12}/a_{12}$ meaning that the first good has a lower revenue-cost ratio in the first process than in the second process.

With one process being less profitable in producing one good but the other process being less profitable in producing the other good, a way would exist in which the two processes could be earning the same internal rate of return, i.e., if both goods are economic: $p > 0$.

Would it at the same time be possible for both processes to be operated?

Suppose $b_{22}/a_{22} < b_{12}/a_{12}$ meaning that in the second process the second good is growing less rapidly than the first good. So if only the second process were operated the second good would be growing less rapidly than the first good.

But also suppose $b_{11}/a_{11} < b_{21}/a_{21}$ meaning that in the first process the first good is growing less rapidly than the second good.

With one good growing less rapidly in one process but the other good growing less rapidly in the other process, a way would exist in which the two goods could be growing at the same rate, i.e., if both processes were operated: $x > 0$.

By a stroke of the pen let us reverse the four less-than signs. Then "lower revenue cost ratio" would become "higher revenue-cost ratio," but it would remain true that one process was less profitable

in producing one good and the other process less profitable in producing the other good, and it would remain true that both goods would have to be economic: $p > 0$. Furthermore "growing less rapidly" would become "growing more rapidly," but it would remain true that one good would be growing less rapidly in one process and the other good growing less rapidly in the other process, and it would remain true that both processes would have to be operated: $x > 0$.

16. Fourth Possibility. No Free Good and No Unused Process: The Algebra

The four conditions using less-than signs would be met if

$$\frac{b_{11}}{a_{11}} < \frac{b_{22}}{a_{22}} < \frac{b_{12}}{a_{12}} < \frac{b_{21}}{a_{21}} \text{ or } \frac{b_{11}}{a_{11}} < \frac{b_{22}}{a_{22}} < \frac{b_{21}}{a_{21}} < \frac{b_{12}}{a_{12}} \text{ or}$$

$$\frac{b_{22}}{a_{22}} < \frac{b_{11}}{a_{11}} < \frac{b_{12}}{a_{12}} < \frac{b_{21}}{a_{21}} \text{ or } \frac{b_{22}}{a_{22}} < \frac{b_{11}}{a_{11}} < \frac{b_{21}}{a_{21}} < \frac{b_{12}}{a_{12}} \qquad (40)$$

and the four conditions using greater-than signs would be met by reversing all less-than signs in (40).

If both goods were economic, $p > 0$, there would be zero excess supply of the first good, $u_1 = 0$. If both processes were operated, $x > 0$, there would be zero loss margin of the first process, $v_1 = 0$.

For $u_1 = v_1 = 0$ would our system of quadratics (27) and (24) have a unique set of feasible solutions for relative price p and relative process level x? With $u_1 = v_1 = 0$ our system collapses into

$$p^2 + Hp + I = 0 \tag{41}$$

$$x^2 + Jx + K = 0 \tag{42}$$

where

$$H \equiv \frac{a_{11}b_{22} - a_{12}b_{21} + a_{21}b_{12} - a_{22}b_{11}}{a_{11}b_{12} - a_{12}b_{11}}$$

$$I \equiv \frac{a_{21}b_{22} - a_{22}b_{21}}{a_{11}b_{12} - a_{12}b_{11}}$$

$$J \equiv \frac{a_{11}b_{22} + a_{12}b_{21} - a_{21}b_{22} - a_{22}b_{11}}{a_{11}b_{21} - a_{21}b_{11}}$$

$$K \equiv \frac{a_{12}b_{22} - a_{22}b_{12}}{a_{11}b_{21} - a_{21}b_{11}}$$

The roots of our quadratics will be

$$p = - H/2 \pm \sqrt{(- H/2)^2 - I} \tag{43}$$

$$x = - J/2 \pm \sqrt{(- J/2)^2 - K} \tag{44}$$

As long as (40) holds with the less-than signs the denominator $a_{11}b_{12} - a_{12}b_{11}$ of H and I as well as the denominator $a_{11}b_{21} - a_{21}b_{11}$ of J and K would be positive. The numerators $a_{21}b_{22} - a_{22}b_{21}$ of I and $a_{12}b_{22} - a_{22}b_{12}$ of K would both be negative. As long as (40) holds after reversal of the inequality signs those denominators and numerators would be negative and positive, respectively. Either way I and K would both be negative, hence -I in (43) and -K in (44) both positive making the absolute value of the square roots of (43) and (44) greater than the absolute value of the first term -H/2 and -J/2, respectively. As a result (43) for p and (44) for x would have one positive (feasible) and one negative (nonfeasible) root regardless of the sign of -H/2 and -J/2, respectively. As long as (40) holds before and after reversal of

inequality signs, then, we have proved the existence of a unique set of feasible solutions for relative price p and relative process level x.

With both goods being economic, what will their common rate of growth be? When we found our quadratic (24) in x we set the right-hand sides of (22) and (23) equal and multiplied across. So we may find our equilibrium maximum rate of growth g by inserting $u_1/X_2 = 0$ and our solution (44) for x into either (22) or (23).

With both processes being operated, what will their common internal rate of return be? When we found our quadratic (27) in p we set the right-hand sides of (25) and (26) equal and multiplied across. So we may find our equilibrium minimum rate of interest r by inserting $v_1/P_2 = 0$ as well as our solution (43) for p into either (25) or (26).

IV. CONCLUSIONS

1. Solution

The best way to prove the existence of a solution is to find one. By reducing the number of goods to two and the number of processes to two, we have succeeded in solving a von Neumann model by using nothing

but elementary algebra--thus needing neither the game theory nor the matrix algebra applied in the excellent restatements by Kemeny, Snell, and Thompson (1957: 353-367) and Lancaster (1968: 164-168).

Since the von Neumann system is homogeneous of degree zero in its process levels, excess supplies, goods prices, and loss margins, we were able to find solutions only for relative process levels, excess supplies, goods prices, and loss margins. Since the numbering of goods and processes is arbitrary we assumed that at least the second good would always become economic and that at least the second process would always be used. Our solutions covered several possibilities.

For the second good to become an economic good its rate of growth must be either less than or equal to the rate of growth of the first good. If the less-than sign applies, the second good will be the only economic good. Such will be the case if in both processes the second good is growing less rapidly than the first good. If the equal-to sign applies, both goods will be economic. Such will be the case if one good is growing less rapidly in one process but the other good growing less rapidly in the other process, and both processes are operated. As a very special case, even if only one process is operated both goods may become economic if growing at the same rate in that process.

For the second process to be operated its internal rate of return must be either greater than or equal to the internal rate of return of

the first process. If the greater-than sign applies, the second process will be the only one operated. Such will be the case if both goods have a lower revenue-cost ratio in the first process than in the second process. If the equal-to sign applies, both processes will be operated. Such will be the case if one process is less profitable in producing one good but the other process less profitable in producing the other good, and both goods are economic. As a very special case, even if only one good is economic both processes may be operated if equally profitable in producing that good.

The heart of the von Neumann model is its saddle point: the maximized rate of growth equals the minimized rate of interest. An economist would make two observations on that saddle point.

2. First Observation: Saving and Consumption

In a growing economy somebody must be saving. We may think of a von Neumann model as having capitalists in it who are lending money capital to the entrepreneurs to carry them over their one-time unit period of production. At the rate of interest r capitalists at the beginning of that period lend the entrepreneurs the sum $a_{11}P_1X_1 + a_{12}P_1X_2 + a_{21}P_2X_1 + a_{22}P_2X_2$ financing the purchases of all

goods absorbed as inputs, where P_1 and/or X_1 may be zero in some cases.

At the end of the period of production what will the value of aggregate output be? Let us find it in two different ways. First, since the product $P_i u_i$ always has one and only one factor equalling zero, we may set (12) and (13) equal to zero, then add them and find the value of aggregate output

$$b_{11}P_1X_1 + b_{12}P_1X_2 + b_{21}P_2X_1 + b_{22}P_2X_2 =$$

$$(1 + g)(a_{11}P_1X_1 + a_{12}P_1X_2 + a_{21}P_2X_1 + a_{22}P_2X_2) \qquad (45)$$

So aggregate input has grown into aggregate output at the rate g. Second, since the product $v_j X_j$ always has one and only one factor equalling zero, we may set (14) and (15) equal to zero, then add them and once again find the value of aggregate output

$$b_{11}P_1X_1 + b_{12}P_1X_2 + b_{21}P_2X_1 + b_{22}P_2X_2 =$$

$$(1 + r)(a_{11}P_1X_1 + a_{12}P_1X_2 + a_{21}P_2X_1 + a_{22}P_2X_2) \qquad (46)$$

So aggregate input has also grown into aggregate output at the rate r. But in the saddle-point solution (17) the maximized rate of growth g equalled the minimized rate of interest r. Consequently (45) and (46) are equal, and out of their sales proceeds the entrepreneurs can pay back with interest the sum they borrowed from the capitalists one time unit earlier--no more, no less--provided, of course, that the sale of their output can be financed. It can if for the next period of production the capitalists lend the entrepreneurs the sum (45) or (46), thus financing new purchases. What the entrepreneurs as a whole are purchasing, the entrepreneurs as a whole are selling.

In this way we may continue. New debt forever pays off old debt with interest, and the aggregate debt is a rising one, rising at the rate r. What makes it all possible is the willingness of the capitalists to save their entire interest earnings.

Thus only labor consumes in the von Neumann model. The entrepreneurs don't consume anything, because their income *qua* entrepreneurs is zero--pure competition and freedom of entry and exit to see that. Capitalists do have an income, but their propensity to consume it is zero.

3. Second Observation: Decay and Subsidies

We have found nonnegative solutions for $1 + g$ and $1 + r$ such as (23), (26), (32), (33), (36), and (39). Notice that growth and interest factors rather than growth and interest rates are nonnegative and that $1 + g \geqq 0$ merely implies $g \geqq - 1$, and that $1 + r \geqq 0$ merely implies $r \geqq - 1$. A good growing at a rate $- 1 < g < 0$ is gradually decaying. But as in radioactive decay there will always be some of it left. A good growing at a rate $g = - 1$ is disappearing completely and at once. A process paying a rate of interest $- 1 < r < 0$ is being subsidized because its revenue falls short of its cost. But some revenue it does have. A process paying a rate of interest $r = - 1$ is being subsidized because it has no revenue at all.

But if entrepreneurs can come up with a combination of a good, say the second, and a process, say the second, in which

$$b_{22} > a_{22} \tag{47}$$

then the second good can be growing at the positive rate g according to (32), and the second process can pay the positive rate of interest r according to (33).

FOOTNOTE

[1]Schumpeter's (1912) theory of interest is validated by von Neumann: in a stationary economy $g = r = 0$. On Schumpeter's theory of interest see Haberler (1951: 72-78). In a letter of December 7, 1983, Haberler is "pretty sure that Schumpeter was not aware that von Neumann validated his theory of interest."

REFERENCES

G. Cassel, Theoretische Sozialökonomie, third edition, Erlangen and Leipzig, 1923; translated from the fifth edition by S. L. Barron as The Theory of Social Economy, New York, 1932.

R. Dorfman, P. A. Samuelson, and R. Solow, Linear Programming and Economic Analysis, New York, 1958, 381-388.

G. Haberler, "Schumpeter's Theory of Interest," Seymour E. Harris (ed.), Schumpeter, Social Scientist, Cambridge, Mass., 1951, 72-78.

J. G. Kemeny, J. L. Snell, and G. L. Thompson Introduction To Finite Mathematics, Englewood Cliffs, N.J., 1957.

T. C. Koopmans, "Economic Growth at a Maximal Rate," Quart. J. Econ., Aug. 1964, 78, 355-394.

K. Menger, "Austrian Marginalism and Mathematical Economics," in J. R. Hicks and W. Weber (eds.), Carl Menger and the Austrian School of Economics, Vienna 1971, 38-60.

K. Lancaster, Mathematical Economics, New York, 1968.

J. von Neumann, "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," Ergebnisse

eines mathematischen Kolloquiums, 8, Leipzig and Vienna, 1937, 73-83; translated by G. Morgenstern (Morton) as "A Model of General Economic Equilibrium," Rev. Econ. Stud., 1945-1946, 13, 1-9. "The printing was terrible," says Weintraub (1983: 13n.). Von Neumann first read his paper at Princeton in 1932.

J. A. Schumpeter, Theorie der wirtschaftlichen Entwicklung, Leipzig, 1912 translated by Redvers Opie as The Theory of Economic Development, Cambridge, Mass., 1934.

L. Walras, Eléments d'économie politique pure, Lausanne, Paris, and Basle 1874-1877, translated by W. Jaffé as Elements of Pure Economics or the Theory of Social Wealth, Homewood, Illinois, 1954.

E. R. Weintraub, "On the Existence of a Competitive Equilibrium: 1930-1954," J. Econ. Lit., March 1983, 21, 1-39.